UDC 531

INVESTIGATION OF TIME-OPTIMAL RESPONSE OF ELECTROMAGNETIC CORRECTION OF A GYROSCOPE

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The electromagnetic correction of a gyroscope is considered on the assumption that its magnetic rotor rotates in a solenoid (see [1]). The control of rotor axis displacements is effected by regulating the amplitude and phase of current in the solenoid windings. The time-optimal response of the control of angular displacement of the rotor axis is investigated. This paper is related to [2,3].

1. In the correction method described in [1] the mechanical moment **M** acting on the rotor is determined as the vector product $\mathbf{M} = \mathbf{N} \times \mathbf{K}$, where **N** is the rotor magnetic moment vector rotating in the equatorial plane, and **K** is the vector of magnetic intensity of the solenoid field. Axes of the moving and fixed coordinate systems are shown in Fig. 1, where the OX'-axis coincides with that of the rotor, $OY'Z'$ is the equatorial plane, and OX is the longitudinal axis of the solenoid. Projections of the magnetic moment vector on the moving and fixed axes are defined as follows:

$$N_x' = 0, \quad N_y' = N_0 \cos \Omega t, \quad N_z' = N_0 \sin \Omega t$$
$$N_x = N_z' \sin \alpha - N_y' \sin \beta \cos \alpha, \quad N_y = N_y' \cos \beta,$$
$$N_z = N_z' \cos \alpha + N_y' \sin \alpha \sin \beta$$

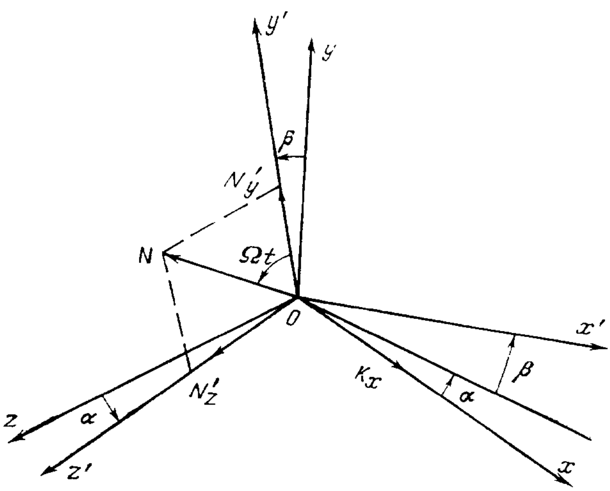

Fig. 1

The vector of magnetic field intensity in the solenoid has only a longitudinal component $K_x = K \cos (\Omega t + \varphi)$ and $K_y = K_z = 0$. The control system is devised so that the field intensity amplitude is subjected to constraint $|K| \leqslant K_0$. By expanding the vector product M we obtain for the components of the mechanical moment the following expressions: $M_x = 0$, $M_y = K_x N_z$, and $M_z = -K_x N_y$. Substituting the expressions for K_x, N_y, and N_z, we obtain for the mechanical correction moments the following formulas:

$$M_y = K \cos(\Omega t + \varphi)\,[N_0 \sin \Omega t \cos \alpha + N_0 \cos \Omega t \sin \alpha \sin \beta] \tag{1.1}$$

$$M_z = -K \cos(\Omega t + \varphi) N_0 \cos \Omega t \cos \beta$$

We restrict the analysis to the precession motion, and for the motion of the gyroscope axis relative to the fixed coordinates we use the system described in [4]. We have

$$\alpha^{\cdot} = -\frac{M_z \cos \alpha}{H \cos \beta}, \qquad \beta^{\cdot} = \frac{M_y}{H \cos \beta} \tag{1.2}$$

The expansion of products $\cos(\Omega t + \varphi) \sin \Omega t$ and $\cos(\Omega t + \varphi) \cos \Omega t$ shows that the expressions for moments in (1.1) contain terms that oscillate at frequency 2Ω, and a term with multipliers $\cos \varphi$, and $\sin \varphi$. In practice usually $1 \ll 2\Omega$, and when integrating (1.2) it is possible to reject, without appreciably affecting accuracy, the high-frequency terms for moments and consider the displacements of the rotor axis as due only to the action of control functions (terms with $\cos \varphi$, and $\sin \varphi$) which are a priori known to be smooth in prolonged (in comparison with $\pi\Omega^{-1}$) intervals of time. The parameter K from (1.1) is linear in the right-hand side of (1.2), hence the problem of time-optimal response contains only a single controlling parameter, namely the phase φ, and the relation $K = K_0$ must be satisfied. Using the notation

$$K_0 N_0 (2H)^{-1} = U, \quad u_1 = U \cos \varphi, \quad u_2 = U \sin \varphi$$

we reduce Eqs. (1.2) to the form

$$\alpha^{\cdot} = u_1 \cos \alpha, \quad \beta^{\cdot} = u_2 \frac{\cos \alpha}{\cos \beta} + u_1 \sin \alpha \operatorname{tg} \beta \tag{1.3}$$

The initial conditions for (1.3) are: $\alpha(0) = 0$ and $\beta(0) = 0$. The control functions are bounded by the constraint $u_1^2 + u_2^2 = U^2$. By regulating the phase of the current in the solenoid windings we control system (1.3): $\operatorname{tg} \varphi = u_2 (u_1)^{-1}$.

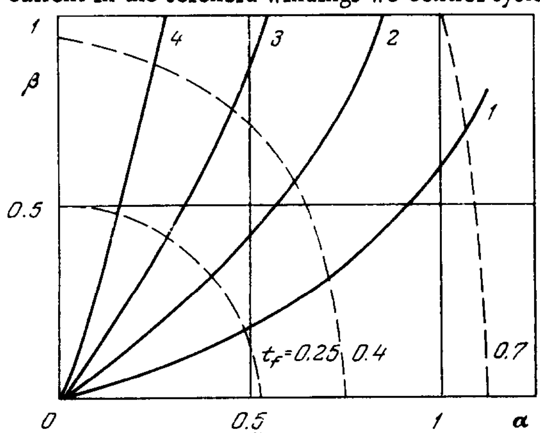

Fig. 2

The problem consists of: 1) analyzing the optimal law of regulating the phase with respect to the time required for bringing system (1.3) from the initial state $\alpha(0)$ and $\beta(0)$ to the specified final state α_f and β_f, in the shortest time t_f taking into consideration the constraint on the control action $u_1^2 + u_2^2 = U^2$, and 2) to determine the quasioptimal law of phase regulation that would actually effect the transition of the system from the initial to the specified point in a time close to optimal.

2. Let us apply the principle of the maximum [5]. The Hamiltonian and the equations for the conjugate system are of the form

$$H = -1 + \psi_1 u_1 \cos\alpha + \psi_2 \left(u_1 \sin\alpha \operatorname{tg}\beta + u_2 \frac{\cos\alpha}{\cos\beta}\right) \tag{2.1}$$

$$\psi_1^{\cdot} = \psi_1 u_1 \sin\alpha - \psi_2 u_2 \operatorname{tg}\beta \cos\alpha + \psi_2 u_2 \frac{\sin\alpha}{\cos\beta} \tag{2.2}$$

$$\psi_2^{\cdot} = -\psi_2 u_1 \frac{\sin\alpha}{\cos^2\beta} - \psi_2 u_2 \frac{\cos\alpha\sin\beta}{\cos^2\beta}$$

It should be noted that for solving the optimization problem using the procedure described in [5] the right-hand sides of (1.3) must be continuous with respect to the control and continuously differentiable with respect to constants of state. For this it is necessary to consider (1.3) for $|\beta| \leqslant \beta^* < \pi/2$, and arbitrary α. After collecting in (2.1) terms with u_1 and u_2, we find that the optimal control that maximizes (2.1) is defined as follows:

$$u_1^\circ(t) = f_1 \|f\|^{-1} U, \quad u_2^\circ(t) = f_2 \|f\|^{-1} U \tag{2.3}$$

$$f = (f_1, f_2), \quad f_1 = \psi_1 \cos\alpha + \psi_2 \sin\alpha \operatorname{tg}\beta, \quad f_2 = \psi_2 \cos\alpha \cos^{-1}\beta$$

Determination of the optimal control that translates the body from the initial to the specified final point in the minimum of time is related to the solution of the boundary value problem for the system of Eqs. (1.3) and (2.2) with allowance for formula (2.3). Since it is hardly possible to obtain $u_1^\circ(t)$ and $u_2^\circ(t)$ in analytic form, we shall investigate the phase trajectories of system (1.3) by solving on a computer the system of Eqs. (1.3), (2.2), and (2.3) the specified initial conditions: $\psi_1(0)$ and $\psi_2(0)$.

Vector $\psi = (\psi_1, \psi_2)$ can be determined to within the constant multiplier. Moreover, the optimal controls $u_1^\circ(t)$ and $u_2^\circ(t)$ depend, as functions of time, only on the ratio ψ_2/ψ_1 (2.3). Becuase of this we specify the initial conditions for the set of conjugate variables as $\psi_{1i} = \cos\theta_i$, and $\psi_{2i} = \sin\theta_i$.

The set of phase trajectories 1 − 4 is shown in Fig. 2 for $\alpha \geqslant 0$, $\beta^* > \beta > 0$ and $U = 2$, $\theta_i = i\Delta\theta$, $\Delta\theta = 0.3$, $i = 1, \ldots, 4$. It may be pointed out that in the linear approximation (1.3) the optimal controls are

$$u_1^\circ(t) = U\alpha_f [\alpha_f^2 + \beta_f^2]^{-1/2}, \quad u_2^\circ(t) = U\beta_f [\alpha_f^2 + \beta_f^2]^{-1/2}$$

and $\psi_1 = \psi_1(0) = u_1^\circ U^{-1}$, and $\psi_2 = \psi_2(0) = u_2^\circ U^{-1}$. When α and β are close to zero the phase trajectories are straight lines with gradients $\operatorname{tg}\theta_i$. It follows from (1.3) that the change of signs of u_1 and u_2 yields phase trajectories that are symmetric about the coordinate origin; if only the sign of u_1 is changed, the phase trajectories are symmetric about the β-axis. The dash lines connect points at which the times t_f of maximum time-optimal response are equal.

Dependence of the correction current phase φ on time is shown in Fig. 3 in the form of trajectories numbered 1 − 4 calculated on a computer. It is seen that the optimal value of phase φ is virtually constant throughout the control operation time. We shall use this feature for constructing a quasi-optimal control that would bring the system to point α_f, β_f with some fixed u_1 and u_2 which ensure the constancy of phase.

3. Let us consider (1.3) on the assumption that u_1 and u_2 are independent of time. The solution of the first of Eqs. (1.3) with allowance for the initial condition

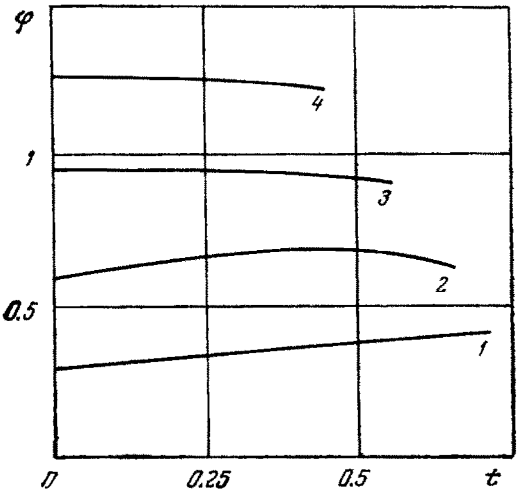

Fig. 3

$\alpha(0) = 0$ is of the form

$$\alpha = -\pi / 2 + 2 \operatorname{arctg} E(t), \quad E(t) = \exp u_1 t \tag{3.1}$$

Then $\cos\alpha$ and $\sin\alpha$ are functions of time of the form

$$\cos\alpha = 2E(t)[1 + E^2(t)]^{-1},$$

$$\sin\alpha = [E^2(t) - 1][1 + E^2(t)]^{-1}$$

After the substitution $d\beta \cos\alpha = dx$ the second of Eqs. (1.3) becomes linear

$$x^{\cdot} + p(x)x = q(t); \quad p(t) = -u_1 \sin\alpha, \; q(t) = u_2 \cos\alpha \tag{3.2}$$

The general solution of (3.2) is of the form

$$x = \{\int q(t)[\exp \int p(t)dt]\, dt + C\} \exp\{-\int p(t)\, dt\}$$

Taking into account that $x(0) = \sin\beta(0) = 0$ and integrating, we obtain for β the expressions

$$\sin\beta = u_2 [E^2(t) - 1][2u_1 E(t)]^{-1}, \quad 0 \leqslant \beta \leqslant \beta^* \tag{3.3}$$

For specified u_1 and u_2 it is necessary to limit the integration time interval to $0 \leqslant t \leqslant t_0$ using the inequality

$$\sin\beta^* > u_2 [E^2(t_0) - 1][2u_1 E(t_0)]^{-1}$$

From (3.1) and (3.3) we obtain the relation of u_1 and u_2 to α_f and β_f

$$\sin\beta_f = u_2 (2u_1)^{-1} [\operatorname{tg}(\alpha_f / 2 + \pi / 4) - \operatorname{ctg}(\alpha_f / 2 - \pi / 4)] \tag{3.4}$$

which can be used for determining u_1 and u_2 for specified coordinates α_f and β_f of the final point (bearing in mind that $u_1^2 + u_2^2 = U^2$). It can be shown that in the case of linear approximation $\alpha_f \to 0$ and $\beta_f \to 0$ the relationship $u_2 u_1^{-1} = \beta_f \alpha_f^{-1}$, follows from (3.4), i.e. for small angles the proposed control becomes the time - optimal response.

Let us estimate the comparative loss of time obtaining with the proposed quasi-optimal control law. Using (3.1) for the time of translation from one point to another in the quasi-optimal motion we obtain

$$t_{f0} = u_1^{-1} \ln \operatorname{tg}(\alpha_f / 2 + \pi / 4)$$

We compare it with the time-optimal response motion at points α_f and β_f lying on isochrones. The solid curves 1 and 2 in Fig. 4 represent the relative losses of response time in percent $\varepsilon_1 = (t_{f0} - t_f) t_f^{-1} \times 100\%$ as functions of β_f for two isochrones $t_f = 0.25$ and $t_f = 0.4$, respectively.

To compare it with the proposed quasi-optimal control we consider a control procedure consisting of, first, bringing the system along the β -axis from zero to β_f with

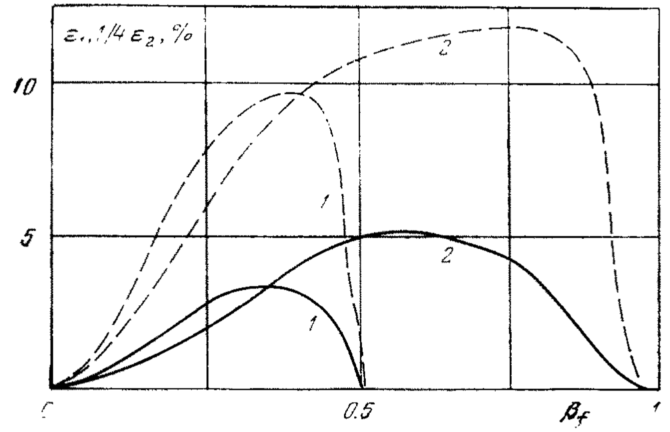

Fig. 4

$u_2 = U$ in the time interval t_1 and, then, convey it form point $(0, \beta_f)$ to point (α_f, β_f) in the time interval t_2 with $u_1 = U$. It can be readily shown that

$$t_1 = U^{-1} \sin \beta_f,\ t_2 = U^{-1} \ln \operatorname{tg} (\alpha_f / 2 + \pi / 4),\ t_{f1} = t_1 + t_2$$

The dash lines in Fig. 4 denote the relative error in terms of β_f $\varepsilon_2 = (t_{f1} - t_f)$ $t_f^{-1} \times 100\%$ for the same isochrones. Comparison of the relative error curves shows that the time-optimal response of the proposed quasi-optimal control differs from that of the optimal by not more than 5%. The second control method, which is fairly simple in execution, results in excesses of the time-optimal response time of up to 45%.

Analysis of the time-optimal response control and of the proposed control, which differs from the optimal, based on (1.3) and curves ε_1 and ε_2 (Fig. 4) shows that, when the coordinates of final points are $(\alpha_f, \beta_f \approx 0)$ and $(\alpha_f \approx 0, \beta_f)$, all of the considered controls are equivalent as regards their time-optimal response.

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